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A three-dimensional parabolic punch problem in linear elasticity

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Abstract. In this paper an analytic solution for a three-dimensional contact problem, in linear elasticity, is constructed through the separation of Laplace's equation in paraboloidal coordinates. A rigid punch under normal loading is applied to an isotropic elastic medium occupying an infinite half-space where the contact region is parabolic and the punch profile is prescribed. This treatment allows for a general punch profile provided it is physically reasonable so as to ensure the convergence of the solution.

1. Introduction

In the theory of elasticity, a punch problem arises when a rigid frictionless punch is pressed against an elastic medium occupying the infinite half-space $z \geqslant 0$. Let S denote the region of contact between the base of the punch and the surface $z = 0$, and $\bar{S}$ the region outside S on $z = 0$. Then, as explained in [4] and [8] (Ch. 1), from the equations of elastostatics for a linearly elastic, homogeneous and isotropic medium (with zero shearing stress on $z = 0$), the problem of determining stresses and displacements in the material is reduced to that of determining a function $\psi(x, y, z)$, harmonic in $z > 0$, satisfying appropriate boundary conditions; in effect,

(i) $\psi|_{z=0}$ is a prescribed function of x, y on S,

(ii) $\left.\dfrac{\partial \psi}{\partial z}\right|_{z=0} = 0$ on $\bar{S}$.

Then the two quantities of greatest physical interest, namely the normal component of stress, $\tau_{zz}(x, y, 0)$, immediately under the punch, and the normal component of displacement, $w(x, y, 0)$, of the surface outside the punch, are given in terms of ψ by

$$w(x, y, 0) = \frac{1-\nu}{\mu}\, \psi(x, y, 0) \quad \text{on } S\,,$$

$$\tau_{zz}(x, y, 0) = \left.\frac{\partial \psi}{\partial z}\right|_{z=0} \quad \text{on } \bar{S}\,.$$

Here, μ, ν denote the shear modulus and Poisson's ratio.

In this paper we shall assume that there is complete contact between the base of the punch and the surface of the elastic medium and that $\tau_{zz}(x, y, 0) = 0$ on $\bar{S}$. In complete contact problems $\tau_{zz}(x, y, 0) < 0$ for all (x, y) in S. For a more detailed discussion the reader is referred to [4] (Sec. 1).

The solution of the problem for a circular punch (of arbitrary punch profile) has long been known. For an elliptic punch, the solution was given by Shail [14] using a classical approach of transforming to ellipsoidal coordinates, in which the mixed boundary-value problem becomes separable and the solution appears in terms of Lamé polynomials.

In [4] the authors treated the problem of an inifite strip punch, on similar lines, using elliptic cylinder coordinates; the solution involves Mathieu functions. The analysis is, naturally, greatly complicated by the fact that the punch extends (theoretically) over an infinite region.

In the present paper the same technique is applied to a parabolic punch, using paraboloidal coordinates. Again, Mathieu functions are used in the solution, with an additional complication which is investigated in Section 4. In the strip punch problem, the general solution involves the product of a trigonometric function and two Mathieu functions [4] (Eq. (3.7)), whereas here we have the product of three Mathieu functions: consequently instead of using a Fourier cosine transform we have to employ a less familiar integral relationship for Mathieu functions.

It may be noted here that crack and punch problems for the same regions can usually be solved along similar lines. This is due to the fact that if such problems are viewed as mixed boundary-value problems of potential theory then by switching the appropriate boundary conditions of one problem we can define the other. In the punch problem we have a state of zero normal stress outside the punch as well as a prescribed function for the normal component of displacement under the punch, whereas in the corresponding crack problem the normal component of displacement outside the crack is zero and the normal component of stress (pressure) is prescribed inside. A solution to the two-dimensional parabolic contact problem is given by England [6], and the parabolic crack problem for uniform pressure has been solved by Shah and Kobayashi [13]. Also in a paper by Kassir [10], solutions for parabolic crack problems under uniform pressure, uniform shear and pure bending are given.

It is worth remarking that the problem for an indentation of uniform depth, the parallel to the uniform-pressure crack problem considered by Shah and Kobayashi, is not physically reasonable because it would require an infinite production of energy. It is not surprising, therefore, that our method of solution does not apply to this case although, of course, an approximation to this situation could be handled.

Our solution to the parabolic punch problem allows a general representation for normal displacements under the punch (within physically reasonable limits) which in terms of the corresponding crack problem amounts to allowing a general pressure distribution inside the crack.

It should be emphasized that the solution we give is not merely theoretical; recent progress in the numerical computation of Mathieu functions brings the solution well within the area where numerical results can be obtained for specific cases. Section 6 of this paper illustrates the solution in one such case.

2. Formulation of the general boundary-value problem

In terms of the Cartesian coordinates (x, y, z) let the contact region S be the interior of a parabolic plate in the xy-plane and let $\bar{S}$ denote the region outside S on $z = 0$ which we assume to be stress-free.

The parabola describing the boundary of S has its vertex at $(\frac{1}{2}c, 0, 0)$ and its axis coincides with the x-axis such that all points with coordinates $(x, 0, 0)$ where $x < \frac{1}{2}c$ lie inside S. More specifically, in terms of the paraboloidal coordinates (α, β, γ) (discussed below), the region S is the degenerate surface corresponding to $\alpha = 0$.

A rigid frictionless punch whose profile is defined by the function $K(x, y)$ is applied to the region S and contact is assumed to be complete everywhere on S. This assumption is highly significant in application of the theory to practical problems. The mathematical analysis which follows can be carried out without any assumption regarding completeness of contact and apparently sensible answers can be obtained; at the end, however, one has to check that the theoretical pressure p between the punch and the surface is everywhere non-negative, and that check must be done in each individual problem. In problems where the contact area is only known qualitatively, it is necessary to follow an iterative procedure – see, for example, Kalker and Van Randen [9]. We shall also assume that $K(x, y)$ is symmetric about $y = 0$. A general profile can be written as the sum of two functions, one symmetric and the other antisymmetric about $y = 0$ and the corresponding solutions can then be superposed. For zero shearing stress on $z = 0$, the general equations of elasticity [4] (Sec. 1) can be used to reduce this problem to a mixed boundary-value problem of potential theory with the following boundary conditions:

$$w(x, y, 0) = K(x, y) \quad \text{on } S\,, \qquad \tau_{zz}(x, y, 0) = 0 \quad \text{on } \bar{S}\,.$$

Thus, a solution to the boundary-value problem may be regarded as the harmonic function ψ which satisfies the following conditions:

(a) $\nabla^2\psi = 0$ for $z > 0$,
(b) $\psi \to 0$ as $R \to \infty$, $(R = (x^2 + y^2 + z^2)^{1/2})$, in $z > 0$,
(c) $\dfrac{\partial \psi}{\partial z} = 0$ on $\bar{S}$,
(d) $\{(1 - \nu)/\mu\}\psi(x, y, 0) = K(x, y)$ on S,
(e) $K(x, y)$ is symmetric about $y = 0$.

In order to solve this problem by the method of separation of variables we shall employ the paraboloidal coordinate system. The paraboloidal coordinates (α, β, γ) are related to the Cartesian coordinates by

$$x = \tfrac{1}{2}c(\cosh 2\alpha + \cos 2\beta - \cosh 2\gamma)\,, \qquad y = 2c \cosh \alpha \cos \beta \sinh \gamma\,,$$
$$z = 2c \sinh \alpha \sin \beta \cosh \gamma\,,$$

where α, β and γ are all real, c is a dimensional parameter and

$$0 \leqslant \alpha < \infty\,, \qquad -\pi < \beta \leqslant \pi\,, \qquad 0 \leqslant \gamma < \infty\,.$$

Since this coordinate system is not well known, a brief description follows.

The surfaces α = constant consist of a family of elliptic paraboloids. In particular, if $\alpha = \alpha_0$ the vertex of the elliptic paraboloid is given by $(\tfrac{1}{2}c \cosh 2\alpha_0, 0, 0)$ (in the Cartesian coordinate system) and its axis is the x-axis so that a point with coordinates $(x, 0, 0)$ where $x \leqslant \tfrac{1}{2}c \cosh 2\alpha_0$ lies inside the elliptic paraboloid. The section of this paraboloid by a plane perpendicular to the x-axis is an ellipse, the sections by the planes $y = 0$ and $z = 0$ are both parabolas.

On such a surface $\alpha = \alpha_0$ there is a singular arc, namely the intersection with the surface $\gamma = 0$; on this singular arc the correspondence between (x, y, z) and (α, β, γ) coordinates ceases to be one-to-one. If $0 < \beta' < \pi/2$, the points with paraboloidal coordinates $(\alpha_0, \frac{1}{2}\pi \pm \beta', 0)$ coincide, also the pair $(\alpha_0, -\frac{1}{2}\pi \pm \beta', 0)$ represent the same point. As shown in [1] this has important consequences if we require a solution of $\nabla^2\psi = 0$ to have continuous gradient across the arc.

For $\alpha = 0$ we obtain the degenerate surface occupied by a parabolic plate in the x, y-plane with vertex at $(\frac{1}{2}c, 0, 0)$. The surfaces $\gamma =$ constant also consist of a family of elliptic paraboloids. For $\gamma = \gamma_0$ we have the elliptic paraboloid with vertex at $(-\frac{1}{2}c \cosh 2\gamma_0, 0, 0)$ whose axis is the x-axis, and points with Cartesian coordinates (x, y, z) where $x \geqslant -\frac{1}{2}c \cosh 2\gamma_0$ lie inside this paraboloid. The sections are similar to those of $\alpha = \alpha_0$. On such a surface $\gamma = \gamma_0$, the intersection with $\alpha = 0$ forms another singular arc, where the points with paraboloidal coordinates $(0, \pm\beta', \gamma_0)$, for $0 < \beta' < \pi/2$, coincide. This also has implications for continuity of solutions and continuity of the gradients of solutions across the arc.

When $\gamma = 0$ we obtain the parabolic plate in the x, z-plane with vertex at $(-\frac{1}{2}c, 0, 0)$. Finally, the surfaces $\beta =$ constant consist of portions of hyperbolic paraboloids. $\beta = \beta_0$ gives one quarter of a hyperbolic paraboloid, and the complete paraboloid is given by $\beta = \pm\beta_0$, $\beta = \pm(\pi - \beta_0)$. The degenerate surfaces are $\beta = 0$, $\pm\pi/2$ and π. For our purposes, the region formed by $\beta = 0$ and $\beta = \pi$ is of interest since it is the infinite plate with a parabolic hole in the x, y-plane which occupies the exterior of the surface $\alpha = 0$ in this plane.

3. Separation of Laplace's equation in paraboloidal coordinates

Laplace's equation $\nabla^2\psi = 0$, in paraboloidal coordinates is given by [2]

$$(\cos 2\beta + \cosh 2\gamma)\frac{\partial^2\psi}{\partial\alpha^2} + (\cosh 2\gamma + \cosh 2\alpha)\frac{\partial^2\psi}{\partial\beta^2} + (\cosh 2\alpha - \cos 2\beta)\frac{\partial^2\psi}{\partial\gamma^2} = 0\,, \tag{3.1}$$

Let $\psi = A(\alpha)B(\beta)C(\gamma)$. Then the separated equations are:

$$A''(\alpha) + (-\lambda + 2q\cosh 2\alpha)A(\alpha) = 0\,, \tag{3.2}$$

$$B''(\beta) + (\lambda - 2q\cos 2\beta)B(\beta) = 0\,, \tag{3.3}$$

$$C''(\gamma) + (-\lambda - 2q\cosh 2\gamma)C(\gamma) = 0\,, \tag{3.4}$$

where λ and $2q$ are separation constants chosen so that (3.3) takes the standard form of Mathieu's equation. Initially λ and q are arbitrary and independent but, as we shall see later, the boundary conditions of our problem will require that q be negative, say $q = -h^2$, where $h \in [0, \infty)$. The separation constant $\lambda = \lambda(h^2)$ turns out to be one of the characteristic values a_n, b_n of Mathieu's equation [4] (Sec. 3).

Arscott [1] called solutions of (3.4) 'co-Mathieu functions'. These solutions can be expressed conveniently in terms of solutions of (3.2), namely the 'modified' Mathieu functions.

With the solutions of equations (3.2)–(3.4) in mind, we quote various needed results from Mathieu-function theory.

Consider the ordinary and modified Mathieu equations, respectively

$$\frac{d^2w}{dz^2} + (\lambda - 2q\cos 2z)w = 0\,, \tag{3.5}$$

$$\frac{d^2w}{dz^2} - (\lambda - 2q\cosh 2z)w = 0\,, \tag{3.6}$$

(3.6) being obtained from (3.5) by changing z to iz.

Equation (3.5) has the same qualitative nature whether q is positive or negative. Indeed, it is easily seen that if $w(z, q)$ is a solution of (3.5), then $w(\frac{1}{2}\pi - z, -q)$ is also a solution. This remark leads to the well known relations between 2π-periodic Mathieu functions of the first kind, where using McLachlan's notation [12] (Sec. 2.18) we have

$$\mathrm{ce}_{2n}\left(\frac{\pi}{2} - z, q\right) = (-1)^n \,\mathrm{ce}_{2n}(z, -q)\,, \tag{3.7a}$$

$$\mathrm{ce}_{2n+1}\left(\frac{\pi}{2} - z, q\right) = (-1)^n \,\mathrm{se}_{2n+1}(z, -q)\,, \tag{3.7b}$$

$$\mathrm{se}_{2n+1}\left(\frac{\pi}{2} - z, q\right) = (-1)^n \,\mathrm{ce}_{2n+1}(z, -q)\,, \tag{3.7c}$$

$$\mathrm{se}_{2n+2}\left(\frac{\pi}{2} - z, q\right) = (-1)^n \,\mathrm{se}_{2n+2}(z, -q)\,. \tag{3.7d}$$

In the problem under consideration, we are interested only in these 2π-periodic Mathieu functions of the first kind. The parameter λ must, of course, have the appropriate characteristic value $a_m(q)$ or $b_m(q)$.

In equation (3.6), on the other hand, a change of sign of q changes the qualitative nature of the equation completely. If $q > 0$, say $q = h^2$, then (at least for sufficiently large z) the coefficient of w is negative, so the equation is oscillatory. The two standard solutions are the modified Mathieu functions of the first and second kind. To be specific, let us take the case where $\lambda = a_{2n}(h^2)$, so that the periodic solution of (3.5) is $\mathrm{ce}_{2n}(z, h^2)$; the solutions of (3.6) are then respectively

$$\mathrm{Ce}_{2n}(z, h^2) = \mathrm{ce}_{2n}(iz, h^2) \quad \text{and} \quad \mathrm{Fey}_{2n}(z, h^2)\,.$$

As $z \to \infty$ these are both oscillatory and tend to zero, their asymptotic behavior being, as $z \to \infty$, [12] (Sec. 11.10)

$$\mathrm{Ce}_{2n}(z, h^2) \sim p_{2n}(h^2)\left(\frac{2}{\pi v}\right)^{1/2} \sin\left(v + \frac{\pi}{4}\right)$$

$$\mathrm{Fey}_{2n}(z, h^2) \sim -p_{2n}(h^2)\left(\frac{2}{\pi v}\right)^{1/2} \cos\left(v + \frac{\pi}{4}\right)$$

where $v = he^z$ and $p_{2n}(h^2) = \mathrm{ce}_{2n}(0, h^2)\,\mathrm{ce}_{2n}(\frac{1}{2}\pi, h^2)/A_0^{(2n)}(h^2)$.

On the other hand, if q is negative, then equation (3.6) is non-oscillatory [5] (Appendix A) and the solutions are exponentially increasing or decreasing. The standard solutions are the modified Mathieu functions of the first and third kind, namely

$$\mathrm{Ce}_{2n}(z, -h^2) = \mathrm{ce}_{2n}(iz, -h^2) \quad \text{and} \quad \mathrm{Fek}_{2n}(z, -h^2)$$

with asymptotic behavior (for large z) [12] (Sec. 11.12)

$$\mathrm{Ce}_{2n}(z, -h^2) \sim (-1)^n p_{2n}(h^2)(2\pi v)^{-1/2}\,\mathrm{e}^{v}$$

and

$$\mathrm{Fek}_{2n}(z, -h^2) \sim (-1)^n p_{2n}(h^2)(2\pi v)^{-1/2}\,\mathrm{e}^{-v}$$

where $v = h\,\mathrm{e}^z$.

Next we consider the 'co-Mathieu equation' (3.4) in the form

$$\frac{\mathrm{d}^2 w}{\mathrm{d}z^2} + (-\lambda - 2q\cosh 2z)w = 0\,. \tag{3.8}$$

It is easily verified that if $w(z, q)$ satisfies (3.5), then $w(\frac{1}{2}\pi + \mathrm{i}z, q)$ satisfies (3.8). Following Arscott's notation [1] we write those solutions corresponding to $\mathrm{ce}_{2n}(z, q)$ as

$$\mathrm{CE}_{2n}(z, q) \overset{\mathrm{d.}}{=} \mathrm{ce}_{2n}(\tfrac{1}{2}\pi + \mathrm{i}z, q) = \mathrm{Ce}_{2n}(z - \tfrac{1}{2}\mathrm{i}\pi, q)\,,$$

$$\mathrm{FEY}_{2n}(z, q) \overset{\mathrm{d.}}{=} \mathrm{Fey}_{2n}(z - \tfrac{1}{2}\mathrm{i}\pi, q)\,,$$

$$\mathrm{FEK}_{2n}(z, q) \overset{\mathrm{d.}}{=} \mathrm{Fek}_{2n}(z - \tfrac{1}{2}\mathrm{i}\pi, q)\,.$$

The qualitative nature of this equation depends also on the sign of q, but in the opposite way to that of equation (3.6). For $q > 0$, (3.8) is non-oscillatory, with CE and FEK the real solutions, CE being exponentially increasing and FEK decaying. As $z \to \infty$,

$$\mathrm{CE}_{2n}(z, h^2) \sim p_{2n}(h^2)(2\pi v)^{1/2}\,\mathrm{e}^{v}$$

and

$$\mathrm{FEK}_{2n}(z, h^2) \sim p_{2n}(h^2)(2\pi v)^{1/2}\,\mathrm{e}^{-v}\,.$$

For $q < 0$, the real solutions are CE, FEY, both oscillatory and decaying. We shall not need these in the problem under discussion.

Finally, we observe that formulae (3.7) yield simple links between the modified Mathieu and the co-Mathieu functions. Writing $(\frac{1}{2}\pi + \mathrm{i}z)$ for z in (3.7a) gives

$$\mathrm{Ce}_{2n}(z, q) = (-1)^n\,\mathrm{CE}_{2n}(z, -q)\,. \tag{3.9}$$

Consequently, it is possible to avoid use of the CE functions altogether. We shall retain them while putting our problem into mathematical terms but then eliminate them in favor of the Ce functions in order to analyze the solution.

4. The general solution of the boundary-value problem

In terms of the paraboloidal coordinates (α, β, γ) we wish to find $\psi = \psi(\alpha, \beta, \gamma)$ such that it satisfies the following set of conditions:

(a′) Equation (3.1) holds for $\alpha \in (0, \infty)$, $\beta \in (0, \pi)$ and $\gamma \in (0, \infty)$,

(b′) $\psi \to 0$ as $\alpha \to \infty$ or $\gamma \to \infty$, for $\beta \in [0, \pi]$,

(c′) $(2c \sinh \alpha \cosh \gamma)^{-1} \dfrac{\partial \psi}{\partial \beta} = 0$ at $\beta = 0$ and $\beta = \pi$, where $\alpha \in (0, \infty)$ and $\psi \in [0, \infty)$,

(d′) $\psi(0, \beta, \gamma) = H(\beta, \gamma)$, where

$$H(\beta, \gamma) \overset{\text{d.}}{=} \{\mu/(1-\nu)\} K(x, y)\,, \quad \beta \in (0, \pi) \quad \text{and} \quad \gamma \in [0, \infty)\,,$$

(e′) $H(\beta, \gamma)$ is symmetric about $\beta = \pi/2$.

Equation (3.3) together with the above conditions ((c′) and (e′)) imply that $B(\beta) = \mathrm{ce}_{2n}(\beta, q)$ ([4], Sec. 3) and of course that $\lambda = a_{2n}(q)$, but with no restriction on the sign of q.

We turn to the question of what solution of (3.4) must be chosen. Consider the part of a surface $\alpha = \alpha_0\,(\neq 0)$ which lies inside the elastic medium. This surface is described as β and γ vary over the ranges $0 \leq \beta \leq \pi$ and $0 \leq \gamma < \infty$ with one-to-one correspondence except on the singular arc given by $\gamma = 0$. Here the points corresponding to the triads $(\alpha_0, \tfrac{1}{2}\pi \pm \beta', 0)$, for $0 < \beta' < \tfrac{1}{2}\pi$, coincide. Now, we naturally require that our ultimate solution ψ should be continuous, with continuous gradient, throughout the interior of the elastic medium. As explained in [1], these continuity requirements lead to the conclusion that if $\psi = A(\alpha)B(\beta)C(\gamma)$ and $B(\beta) = \mathrm{ce}_{2n}(\beta, q)$ then we must have $C(\gamma) = \mathrm{CE}_{2n}(\gamma, q)$; the solutions $\mathrm{FEY}_{2n}(\gamma, q)$ and $\mathrm{FEK}_{2n}(\gamma, q)$ are ruled out.

We still have no criterion for the sign of q, but this appears when we take account of condition (b′), for as noted above, $\mathrm{CE}_{2n}(\gamma, q) \to 0$ as $\gamma \to \infty$ only if $q < 0$. We therefore set

$$q = -h^2$$

and our separated solution is of the form

$$\psi = A(\alpha)\, \mathrm{ce}_{2n}(\beta, -h^2)\, \mathrm{CE}_{2n}(\gamma, -h^2)\,.$$

Finally, consider $A(\alpha)$ which satisfies (3.2) with $q = -h^2$. Hence, $A(\alpha)$ may involve $\mathrm{Ce}_{2n}(\alpha, -h^2)$ or $\mathrm{Fek}_{2n}(\alpha, -h^2)$, but the former must be excluded because of condition (b′); as $\alpha \to \infty$, $\mathrm{Ce}_{2n}(\alpha, -h^2) \to \infty$.

So $A(\alpha) = \mathrm{Fek}_{2n}(\alpha, -h^2)$ and our separated solution is necessarily of the form

$$\psi = \mathrm{Fek}_{2n}(\alpha, -h^2)\, \mathrm{ce}_{2n}(\beta, -h^2)\, \mathrm{CE}_{2n}(\gamma, -h^2)\,.$$

More generally, a single separated solution can be written as

$$\psi_n \overset{\text{d.}}{=} \psi_n(\alpha, \beta, \gamma, h) = B_n(h)\, \mathrm{Fek}_{2n}(\alpha, -h^2)\, \mathrm{ce}_{2n}(\beta, -h^2)\, \mathrm{Ce}_{2n}(\gamma, h^2)$$

where $\mathrm{CE}_{2n}(\gamma, -h^2)$ has for convenience been replaced by $\mathrm{Ce}_{2n}(\gamma, h^2)$ and, as in [4] (Sec. 3), n is an arbitrary non-negative integer, h is an arbitrary non-negative parameter and $B_n(h)$ an arbitrary constant. Consequently, a general solution for the mixed boundary-value problem can be written in the form

$$\psi = \int_0^\infty \sum_{n=0}^{\infty} B_n(h)\, \mathrm{Fek}_{2n}(\alpha, -h^2)\, \mathrm{ce}_{2n}(\beta, -h^2)\, \mathrm{Ce}_{2n}(\gamma, h^2)\, \mathrm{d}h\,. \tag{4.1}$$

In the rest of this section, while finding the coefficients $B_n(h)$, we proceed formally; validity is discussed in Section 5.

From boundary condition (d′),

$$H(\beta, \gamma) = \int_0^\infty \sum_{n=0}^\infty C_n(h)\,\mathrm{ce}_{2n}(\beta, -h^2)\,\mathrm{Ce}_{2n}(\gamma, h^2)\,\mathrm{d}h \tag{4.2}$$

where $C_n(h) = B_n(h)\,\mathrm{Fek}_{2n}(0, -h^2)$.

Since $\mathrm{ce}_{2n}(\beta, -h^2) = (-1)^n\,\mathrm{ce}_{2n}(\frac{1}{2}\pi - \beta, h^2)$,

$$H(\beta, \gamma) = \int_0^\infty \sum_{n=0}^\infty (-1)^n C_n(h)\,\mathrm{ce}_{2n}(\tfrac{1}{2}\pi - \beta, h^2)\,\mathrm{Ce}_{2n}(\gamma, h^2)\,\mathrm{d}h\,. \tag{4.3}$$

We now have to invert this relationship in order to obtain $C_n(h)$ in terms of $H(\beta, \gamma)$. Our method involves the use of an integral relationship due to McLachlan [12] (Sec. 10.51, (9)) which in turn is derived from Whittaker's general solution of Laplace's equation; this converts (4.3) into a double Fourier cosine transform.

From [12] (Sec. 10.51, (9)),

$$\mathrm{ce}_{2n}(\tfrac{1}{2}\pi - \beta, h^2)\;\mathrm{Ce}_{2n}(\gamma, h^2) = \rho_{2n} \int_0^{2\pi} \cos[F(\beta, \gamma, \theta, h)]\,\mathrm{ce}_{2n}(\theta, h^2)\,\mathrm{d}\theta$$

where $F(\beta, \gamma, \theta, h) = 2h(\cosh\gamma\,\sin\beta\,\cos\theta + \sinh\gamma\,\cos\beta\,\sin\theta)$, and

$$\rho_{2n} = \mathrm{ce}_{2n}(0, h^2)\,\mathrm{ce}_{2n}(\tfrac{1}{2}\pi, h^2)/2\pi A_0^{(2n)}(h^2)\,.$$

So we can write

$$\begin{aligned} \mathrm{ce}_{2n}(\tfrac{1}{2}\pi - \beta, h^2)\,\mathrm{Ce}_{2n}(\gamma, h^2) = 4\rho_{2n} \int_0^{\pi/2} & \cos(2h\cosh\gamma\,\sin\beta\,\cos\theta) \\ & \times \cos(2h\sinh\gamma\,\cos\beta\,\sin\theta)\,\mathrm{ce}_{2n}(\theta, h^2)\,\mathrm{d}\theta \end{aligned} \tag{4.4}$$

and

$$\begin{aligned} H(\beta, \gamma) = \int_0^\infty \sum_{n=0}^\infty D_n(h) \int_0^{\pi/2} & \cos(2h\cosh\gamma\,\sin\beta\,\cos\theta) \\ & \times \cos(2h\sinh\gamma\,\cos\beta\,\sin\theta)\,\mathrm{ce}_{2n}(\theta, h^2)\,\mathrm{d}\theta\,\mathrm{d}h \end{aligned} \tag{4.5}$$

where $D_n(h) = (-1)^n 4\rho_{2n} C_n(h)$. We interchange the order of summation and integration inside (4.5), then

$$\begin{aligned} H(\beta, \gamma) = \int_0^\infty \int_0^{\pi/2} & \cos(2h\cosh\gamma\,\sin\beta\,\cos\theta) \\ & \times \cos(2h\sinh\gamma\,\cos\beta\,\sin\theta) f(\theta, h)\,\mathrm{d}\theta\,\mathrm{d}h \end{aligned} \tag{4.6}$$

where

$$f(\theta, h) = \sum_{n=0}^\infty D_n(h)\,\mathrm{ce}_{2n}(\theta, h^2)\,. \tag{4.7}$$

Next, we make the following transformations:

$$\xi_1 = h\cos\theta\,, \qquad \xi_2 = h\sin\theta\,, \qquad x_1 = 2\cosh\gamma\sin\beta \quad\text{and}\quad x_2 = 2\sinh\gamma\cos\beta\,,$$

where $\theta\in[0,\tfrac{1}{2}\pi]$, $h\in[0,\infty)$, $\gamma\in[0,\infty)$, $\beta\in[0,\pi]$, so $\xi_1\in[0,\infty)$, $\xi_2\in[0,\infty)$, $x_1\in[0,\infty)$ and $x_2\in(-\infty,\infty)$. Then (4.6) becomes

$$H_1(x_1, x_2) = \int_0^\infty\int_0^\infty \cos(x_1\xi_1)\cos(x_2\xi_2)g(\xi_1, \xi_2)\,d\xi_1\,d\xi_2 \tag{4.8}$$

where $H_1(x_1, x_2) \stackrel{\text{d.}}{=} H(\beta, \gamma)$ and $g(\xi_1, \xi_2) \stackrel{\text{d.}}{=} f(\theta, h)/h$. [From (4.7) and the subsequent expression (4.11) for $D_n(h)$ it can be shown that $f(\theta, h)/h$ is bounded as $h\to 0$.]

Using the two-dimensional Fourier cosine transform formula on (4.8) we get

$$g(\xi_1, \xi_2) = \frac{4}{\pi^2}\int_0^\infty\int_0^\infty \cos(x_1\xi_1)\cos(x_2\xi_2)H_1(x_1, x_2)\,dx_1\,dx_2\,.$$

Changing back to variables h and θ,

$$\sum_{n=0}^\infty D_n(h)\,\mathrm{ce}_{2n}(\theta, h^2) = \frac{4h}{\pi^2}\int_0^\infty\int_0^\infty \cos(x_1 h\cos\theta)\cos(x_2 h\sin\theta)H_1(x_1, x_2)\,dx_1\,dx_2\,. \tag{4.9}$$

Next we multiply both sides of (4.9) by $\mathrm{ce}_{2m}(\theta, h^2)$, where m is a fixed non-negative integer, and integrate with respect to θ from 0 to $\tfrac{1}{2}\pi$. Reversing the order of summation and integration and using the orthogonality of $\mathrm{ce}_{2n}(\theta, h^2)$ we obtain

$$\begin{aligned}\frac{\pi}{4}D_m(h) = \int_0^{\pi/2}\frac{4h}{\pi^2}\,\mathrm{ce}_{2m}(\theta, h^2)\int_0^\infty\int_0^\infty &\cos(x_1 h\cos\theta)\cos(x_2 h\sin\theta)\\ &\times H_1(x_1, x_2)\,dx_1\,dx_2\,d\theta\,.\end{aligned} \tag{4.10}$$

Since $dx_1\,dx_2 = 2(\cosh 2\gamma + \cos 2\beta)\,d\beta\,d\gamma$, then in terms of β and γ, (4.10) can be written as

$$\begin{aligned}D_m(h) = \frac{32h}{\pi^3}\int_0^{\pi/2}\mathrm{ce}_{2m}(\theta, h^2)\int_0^\infty\int_0^{\pi/2} &H(\beta, \gamma)\cos(2h\cosh\gamma\sin\beta\cos\theta)\\ &\times\cos(2h\sinh\gamma\cos\beta\sin\theta)(\cosh 2\gamma + \cos 2\beta)\,d\beta\,d\theta\,.\end{aligned}$$

Finally, interchanging orders of integration and using (4.4), we get

$$\begin{aligned}D_m(h) = \frac{8h}{\pi^3\rho_{2m}}\int_0^\infty\int_0^{\pi/2} &(\cosh 2\gamma + \cos 2\beta)\,\mathrm{ce}_{2m}\left(\frac{\pi}{2} - \beta, h^2\right)\\ &\times\mathrm{Ce}_{2m}(\gamma, h^2)H(\beta, \gamma)\,d\beta\,d\gamma\,.\end{aligned} \tag{4.11}$$

Hence

$$C_n(h) = \frac{2h}{\pi^3\rho_{2n}^2}\int_0^\infty\int_0^{\pi/2}(\cosh 2\gamma + \cos 2\beta)\,\mathrm{ce}_{2n}(\beta, -h^2)\,\mathrm{Ce}_{2n}(\gamma, h^2)H(\beta, \gamma)\,d\beta\,d\gamma \tag{4.12}$$

and the inversion of the relationship (4.3) is complete.

It may be noted here that from (4.1) the normal component of stress under the punch, i.e., on S, is given by $\partial\psi/\partial z|_{z=0}$, namely

$$(2c \sin \beta \cosh \gamma)^{-1} \int_0^\infty \sum_{n=0}^\infty B_n(h)\, \mathrm{Fek}'_{2n}(0, -h^2)\, \mathrm{ce}_{2n}(\beta, -h^2)\, \mathrm{Ce}_{2n}(\gamma, h^2)\, \mathrm{d}h \tag{4.13}$$

where $\beta \in (0, \pi)$ and $\gamma \in [0, \infty)$.

Concerning the edge of the contact region, i.e., where $\alpha = 0$ and $\beta = 0$ or π, the presence of the term $(2c \sin \beta \cosh \gamma)^{-1}$ in (4.13) indicates a square-root type singularity of the function representing the normal component of stress. If $(0, \beta, \gamma)$ are the paraboloidal coordinates of a point inside S and d represents the (shortest) distance from this point to the boundary of S, then

$$d^2 = 4c^2(1 - \cos \beta)^2 \sinh^2\gamma + \frac{c^2}{4}(1 - \cos 2\beta)^2$$

and

$$\sqrt{d} \sim 2c \sin \beta \cosh \gamma f(\gamma) \quad \text{as } \beta \to 0$$

where $f(\gamma) = \frac{1}{2}(c \cosh \gamma)^{-1/2}$. We observe that this type of singularity is expected in complete contact problems [8].

5. On the validity of the formal solution

For a given profile function $H(\beta, \gamma)$ one can examine the corresponding relationships derived in the previous section and determine whether the formal steps are justified or not. It is also possible to specify a set of sufficient conditions to be imposed on the function $H(\beta, \gamma)$ in order to justify the results obtained in Section 4. For the case of the strip-punch problem a detailed discussion is provided in [4] (Sec. 4). Here, however, we shall only outline the main steps of a similar analysis while noting that for the parabolic punch problem, one expects the conditions on H to be somewhat stricter than those stated in [4] for the strip punch. The parabolic punch not only extends to infinity along the x-axis but is also opening out to an infinite width.

To justify the expression (4.3), let

$$T(\xi_1, \xi_2) = \int_0^\infty \int_0^\infty \cos(x_1\xi_1) \cos(x_2\xi_2) H_1(x_1, x_2)\, \mathrm{d}x_1\, \mathrm{d}x_2 \tag{5.1}$$

where ξ_1, ξ_2, x_1, x_2 and $H_1(x_1, x_2)$ are as defined in Section 4. Some conditions on $H_1(x_1, x_2)$ will be required here to ensure the existence of $T(\xi_1, \xi_2)$. Next let $\xi_1 = h \cos \theta$, $\xi_2 = h \sin \theta$ and

$$T(\xi_1, \xi_2) \overset{\text{d.}}{=} \frac{1}{h} J(h, \theta)\,.$$

If we expand $J(h, \theta)$ as a Mathieu function series, under similar conditions to those given in [4] (Sec. 4), we obtain

$$J(h, \theta) = \sum_{n=0}^\infty E_n(h)\, \mathrm{ce}_{2n}(\theta, h^2)$$

where, provided the series is uniformly convergent,

$$E_n(h) = \frac{4}{\pi}\int_0^{\pi/2} J(h, \theta)\, \mathrm{ce}_{2n}(\theta, h^2)\, d\theta .$$

Now we choose the coefficients $C_n(h)$ in (4.3) such that

$$C_n(h) = \frac{(-1)^n E_n(h)}{\pi^2 \rho_{2n}}$$

where ρ_{2n} is defined in Section 4.

It may be noted here that since

$$\mathrm{ce}'_{2n}(0, h^2) = \mathrm{ce}'_{2n}(\tfrac{1}{2}\pi, h^2) = 0$$

and the zeros of basically periodic solutions of Mathieu's equation are all simple, then

$$\mathrm{ce}_{2n}(0, h^2) \neq 0 \quad \text{and} \quad \mathrm{ce}_{2n}(\tfrac{1}{2}\pi, h^2) \neq 0\,, \quad \text{i.e.,} \quad \rho_{2n} \neq 0\,.$$

Finally inverting the double Fourier cosine transform (5.1) we obtain

$$H_1(x_1, x_2) = \frac{4}{\pi^2}\int_0^\infty\int_0^\infty \cos(x_1\xi_1)\cos(x_2\xi_2)\,T(\xi_1, \xi_2)\, d\xi_1\, d\xi_2$$

and by changing the variables back to h, θ, β and γ, we get (4.3). Some further conditions must be imposed on H_1 to ensure that the double Fourier transform can be inverted. For example we can require H_1 to be three times continuously differentiable with respect to x_1 and x_2.

In order to show that the function ψ represented by (4.1) is the solution of the boundary-value problem it must be shown that ψ is continuous and satisfies Laplace's equation together with the boundary conditions. Writing ψ in the form

$$\psi(\alpha, \beta, \gamma) = \int_0^\infty \sum_{n=0}^{\infty} \frac{(-1)^n}{\pi^2} E_n(h) \frac{\mathrm{Fek}_{2n}(\alpha, -h^2)}{\mathrm{Fek}_{2n}(0, -h^2)} \frac{\mathrm{ce}_{2n}(\beta, -h^2)\,\mathrm{Ce}_{2n}(\gamma, h^2)}{\rho_{2n}}\, dh\,, \tag{5.2}$$

we observe that, from [5] (Appendix B),

$$\left|\frac{\mathrm{Fek}_{2n}(\alpha, -h^2)}{\mathrm{Fek}_{2n}(0, -h^2)}\right| \leq 1\,,$$

and from (4.4) and [5] (Appendix C (c. 1.9)),

$$\left|\frac{\mathrm{ce}_{2n}(\beta, -h^2)\,\mathrm{Ce}_{2n}(\gamma, h^2)}{\rho_{2n}}\right| \leq 2\pi(\gamma_0 + \gamma_1 h + \gamma_2 h^2)$$

where γ_0, γ_1, and γ_2 are constants.

Moreover, by the same technique as that used in [5] (Appendix D), a suitable bound can be found for $E_n(h)$ to ensure the uniform convergence of the series and the integral in (5.2). The rest of the analysis, required to demonstrate that ψ is twice differentiable with respect to the three variables, can be developed by modifying the techniques used in [5] (Appendices B, C, and D).

6. An example

We shall now consider an example where the function representing the punch profile, in paraboloidal coordinates, is given by

$$H(\beta, \delta) = \delta[1 + (c^2/4)(\cos 2\beta - \cosh 2\delta)^2]^{-1} \tag{6.1}$$

with δ and c as dimensional parameters. The contact region on the x, y-plane is bounded by the parabola $y^2 = 2c^2 - 4cx$ whose vertex in Cartesian coordinates is at $(c/2, 0, 0)$, and δ is equal to the maximum depth of the punch on the positive z-axis.

This example has been chosen because it has a mathematical form which makes the complete analytical solution possible. Nevertheless it is a physically realistic profile since its longitudinal cross-section which is represented by

$$z = \delta[1 + (x - c/2)^2]^{-1}$$

shows the necessary approach to zero of the punch depth.

The coefficient $B_n(h)$, in the general solution (4.1), can be written as

$$B_n(h) = C_n(h)/\mathrm{Fek}_{2n}(0, -h^2)$$

where $C_n(h)$ is given by (4.12). In order to evaluate the normal component of stress, τ_{zz}, under the punch, $C_n(h)$ together with the various Mathieu functions involved must be calculated. The following analysis demonstrates how $C_n(h)$ can be simplified to a point where its numerical evaluation becomes an easy task. We also note that the expression for $\tau_{zz}(0, \beta, \gamma)$ is

$$(2c \sin\beta \cosh\gamma)^{-1} \int_0^\infty \sum_{n=0}^\infty C_n(h) V_n(h)\, \mathrm{ce}_{2n}(\beta, -h^2)\, \mathrm{Ce}_{2n}(\gamma, h^2)\, \mathrm{d}h \tag{6.2}$$

where $V_n(h) = \mathrm{Fek}'_{2n}(0, -h^2)/\mathrm{Fek}_{2n}(0, -h^2)$.

Starting with (4.10) where $D_n(h) = (-1)^n 4\rho_{2n} C_n(h)$ and $H_1(x_1, x_2) = \delta[1 + (c^2/4)(x_1^2 + x_2^2)^2/4]^{-1}$, we can write $C_n(h)$ as

$$\begin{aligned}&((-1)^n 4\delta h/\rho_{2n}\pi^3) \int_0^{\pi/2} \mathrm{ce}_{2n}(\theta, h^2) \int_0^\infty \int_0^\infty \cos(x_1 h \cos\theta) \cos(x_2 h \sin\theta)\\ &\quad \times [1 + (c^2/4)(x_1^2 + x_2^2)^2/4]^{-1}\, \mathrm{d}x_1\, \mathrm{d}x_2\, \mathrm{d}\theta\,. \end{aligned}\tag{6.3}$$

Next let $t = h\cos\theta$ in (6.3) and use a Fourier cosine transform inversion [7] to obtain

$$\begin{aligned}C_n(h) &= ((-1)^n 8\delta h \mathrm{i}/\rho_{2n}\pi^2 c) \int_0^{\pi/2} \mathrm{ce}_{2n}(\theta, h^2) \int_0^\infty \cos(x_2 h \sin\theta)\\ &\quad \times [(x_2^2 + 4\mathrm{i}/c)^{-1/2}\, \mathrm{e}^{-t(x_2^2+4\mathrm{i}/c)^{1/2}} - (x_2^2 - 4\mathrm{i}/c)^{-1/2}\, \mathrm{e}^{-t(x_2^2-4\mathrm{i}/c)^{1/2}}]\, \mathrm{d}x_2\, \mathrm{d}\theta\,.\end{aligned}$$

The above expression can be inverted one more time, by replacing $h \sin\theta$ by s and noting that $s^2 + t^2 = h^2$. Then

$$C_n(h) = ((-1)^n 8\delta h\mathrm{i}/\rho_{2n}\pi^2 c)\int_0^{\pi/2} \mathrm{ce}_{2n}(\theta, h^2)K(h, c)\,\mathrm{d}\theta$$

where

$$K(h, c) = [K_0((1+\mathrm{i})h(2/c)^{1/2}) - K_0((1-\mathrm{i})h(2/c)^{1/2})]$$

and K_0 is the modified K-Bessel function of order zero.

Finally, since the Mathieu function $\mathrm{ce}_{2n}(\theta, h^2)$ can be represented by a Fourier series expansion, namely

$$\begin{aligned} &\mathrm{ce}_{2n}(\theta, h^2) = \sum_{r=0}^{\infty} A_{2r}^{(2n)}(h^2)\cos 2r\theta\,, \\ &C_n(h) = ((-1)^n 4\delta h\mathrm{i}A_0^{(2n)}(h^2)/\rho_{2n}\pi c)K(h, c)\,. \end{aligned} \tag{6.4}$$

Now given β, γ, and c, the normal component of stress can be calculated by a method similar to that used in [4]. For instance if $c = \frac{1}{2}$, $\beta = \frac{1}{2}\pi$ and $\gamma = 0$, after some simplifications,

$$\tau_{zz}(0, \pi/2, 0) = 16\delta\mathrm{i}\int_0^{\infty}\sum_{n=0}^{\infty} h(A_0^{(2n)})^2 V_n(h)y_n(h)K(h, 1/2)\,\mathrm{d}h \tag{6.5}$$

where $y_n(h) = \mathrm{ce}_{2n}(0, h^2)/\mathrm{ce}_{2n}(\pi/2, h^2)$.

Using small values of h, say $h = 0(0.1)2.0$, and $n = 0, 1, 2$, as in [4] (Sec. 5), τ_{zz} can be evaluated fairly accurately. This is due to the occurrence of the coefficient $A_0^{(2n)}$ and the function $K(h, 1/2)$ in (6.5).

Values of $A_0^{(2n)}(h^2)$ may be obtained from Table 1a ([4]), $V_n(h)$ are tabulated in Table 2 ([4]) and $y_n(h)$ are given in Table 1 of the Appendix. The latter were computed using the method outlined in [3]. In addition, Table 1 of the Appendix contains values of $K(h, 1/2)$ which were obtained from the application of the Romberg procedure with Macsyma ([11]) to an integral representation of the modified K-Bessel functions, namely

$$\mathrm{i}K(h, c) = 2\int_0^{\infty} \mathrm{e}^{-h(2/c)^{1/2}\cosh\theta}\sin(h(2/c)^{1/2}\cosh\theta)\,\mathrm{d}\theta\,.$$

This method has led to the following approximation of τ_{zz},

$$\tau_{zz}(0, \pi/2, 0) \approx \delta(-1.44189)\,.$$

Also considering the point at the boundary of the punch which coincides with the vertex of the parabola ($\alpha = \beta = \gamma = 0$) and excluding the factor $1/\sin\beta$ in (6.2) we obtain the approximation

$$\int_0^{\infty}\sum_{n=0}^{\infty} C_n(h)V_n(h)\,\mathrm{ce}_{2n}(0, -h^2)\,\mathrm{Ce}_{2n}(0, h^2)\,\mathrm{d}h \approx \delta(-1.92272)\,.$$

This result is in accordance with the expected square-root type singularity of the normal component stress in complete contact problems.

7. Parabolic crack problems

Parabolic crack problems which involve nonuniform pressure distributions can be treated by the methods employed in this paper. However, as indicated in [4] (Sec. 6), the boundary conditions will be reversed in the sense that here the pressure, $p(x, y)$, is precribed on S and the normal component of displacement is zero on $\bar{S}$.

Appendix

Table 1. $y_n(h) = \dfrac{\mathrm{ce}_{2n}(0, h^2)}{\mathrm{ce}_{2n}(\pi/2, h^2)}$, $K(h, c) = K_0((1+\mathrm{i})h(2/c)^{1/2}) - K_0((1-\mathrm{i})h(2/c)^{1/2})$

h	$y_0(h)$	$y_1(h)$	$y_2(h)$	$K(h, 1/2)$
0	1.000000	−1.000000	1.000000	1.570796
0.1	0.990037	−1.005013	1.000660	1.475598
0.2	0.960789	−1.013421	1.002613	1.298651
0.3	0.917613	−1.030360	1.006008	1.097885
0.4	0.852414	−1.053046	1.010727	0.898734
0.5	0.779459	−1.086020	1.016819	0.714555
0.6	0.699434	−1.124616	1.024522	0.552024
0.7	0.616466	−1.170027	1.033283	0.413698
0.8	0.534469	−1.221070	1.043761	0.299572
0.9	0.456652	−1.275737	1.055812	0.208084
1.0	0.385434	−1.322370	1.069548	0.136813
1.1	0.322113	−1.384196	1.085111	0.082941
1.2	0.267169	−1.430558	1.102620	0.043578
1.3	0.220388	−1.466529	1.122281	0.015964
1.4	0.181086	−1.488435	1.144303	−0.002405
1.5	0.148421	−1.493723	1.168953	−0.013718
1.6	0.121442	−1.480541	1.196521	−0.019820
1.7	0.099282	−1.448273	1.227318	−0.022225
1.8	0.081114	−1.397352	1.261630	−0.022136
1.9	0.066258	−1.329448	1.299718	−0.020483
2.0	0.054128	−1.237090	1.341702	−0.017959

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